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Decoherent Histories Approach:

A Quantum Description of Closed Systems

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Abstract

The decoherent histories (DH) approach, also known as the consistent histories (CH) approach, was developed in the late 1980s and early 1990s. Introduced over three decades ago, many ideas and variations related to the formulation have been introduced. There were much debates over its interpretations. Aimed at providing new insights and innovations into the formulation and possible theories in which this formulation, particularly no other already developed formulations, of quantum physics is applicable, this paper provides a comprehensive review on the various mathematical frameworks, applications, and interpretations together with critiques on these interpretations of this formulation of quantum physics.

1 Introduction

In 1925 and subsequent years, Heisenberg, Born and Jordan [1], Schrödinger [2] published their groundbreaking papers on the theory of non-relativistic quantum physics known today. The philosophy upon Heisenberg's formulation is that an appropriate quantum theory should only focus on observable quantities. Heisenberg, Born and Jordan's formulation expresses transition amplitudes of an observable as the elements of a matrix. In their formulation, the observed values of the observable is obtained by transformation of the matrix to a diagonal representation. The diagonal elements of the transformed matrix are the possible measurement results. For any pairs of conjugate variables $[p, q]$, they derived the quantisation condition $[p, q] = -i\hbar$ for conjugate variables.

On the other hand, Schrödinger's formulation was set out to look for a wave equation, well known as the Schrödinger equation, for de Broglie's wave. [3] In his paper, Schrödinger also showed that the energy measurement results are the eigenvalues of the Hamiltonian operator of the quantum system. Their formulation successfully recovered the known quantum physical phenomenon at the time. Thence their theory became the standard way for many non-relativistic quantum mechanical computations. It is important to notice that in Heisenberg's picture, the matrix elements represents amplitudes for all possible transitions. So, under the influence of the Hamiltonian it is the matrix elements that changes; whereas in Schrödinger's picture, it is the wavefunction that changes under the influence of the Hamiltonian. From Schrödinger's equation, the time evolution of the wavefunction is given by

$$\Psi(t) = U(t, t')\Psi(t') \tag{1.1}$$

where $U(t, t') = e^{-i\hat{H}(t'-t)}$. Equivalently, one can impose time dependence on matrices, or abstractly on operators, in Heisenberg's picture, by

$$\hat{\mathcal{O}}(t) = U(t, t')\hat{\mathcal{O}}(t')U(t', t) \tag{1.2}$$

The two formulations were later shown to be equivalent [4] as long as one makes appropriate identifications between a Heisenberg operator and Schrödinger operator.

The two formulations were then unified into a single framework, in which the system is represented by a Hilbert space S. [5] The wavefunction of a system is represented by a vector $|\Psi\rangle$ in the Hilbert space. Measurements on the system are represented by transformation of the state by self-adjoint operators: $|\Psi\rangle \mapsto \hat{\mathcal{O}}|\Psi\rangle$. The diagonalisation process is represented by solving for eigenvectors of the operator. Then in the basis of eigenvectors, the operator is represented by a diagonal matrix, with the diagonal elements being the eigenvalues, consistent with the results by Schödinger. The description of the system is represented by a linear combination of the eigenvectors, with the coefficients being the amplitude of obtaining the corresponding eigenvalue upon measurements. The transition amplitude from a general state $|\Psi\rangle$ into an eigenstate $|m\rangle$ upon the measurement of observable $\hat{O}$ is represented by the value $\langle m|\hat{\mathcal{O}}|\Psi\rangle$. The probability of such transition is the squared magnitude of this amplitude. The process of the transformation from $|\Psi\rangle$ to $|m\rangle$ is called a *wavefunction collapse*.

Despite the success of the formulation of Heisenberg, Born and Jordan and Schrodinger in reproducing experimental results, they lead to great debate over the interpretation of the quantum realm. In 1935, Einstein, Podolsky and Rosen published the EPR paper [6] which argued the standard quantum theory cannot be complete. Their argument was that if the wavefunction of the system is a physical description of the system, then there will inevitably an "action over distance" when correlated systems are considered. In response to this, Bohr [7] argued that standard quantum theory is not a realism theory in the sense that any statements about physical properties of a system are meaningless before measurement due to irreversible perturbation a measurement produces on the system. In general, measurement on a physical property $\mathcal{O}$ produces perturbation on other properties $\mathcal{O}'$ unless their associated operators commute. Thence any statements on $\mathcal{O}_1$ becomes meaningless after measurements on $\mathcal{O}_2$ in general.

Putting philosophic discussion aside, a theory with no realism interpretation may seem satisfactory when one is dealing with open systems, i.e. ones that can interact with other parts of a larger system, because physical properties about the open system can be gathered by interactions (measurements). However, this is not so for closed systems. As mentioned above, without the possibility of any measurements, no meaningful statement about the properties of a closed system can be given. As Griffiths stressed [8, 9], the opposite of a meaningless statement is equally meaningless, hence neither predictions or references to the past can be done about closed system with a non-realism theory. This causes great obstacle for a quantum theory of cosmology which is essentially a theory of a closed system.

Another complication of standard quantum mechanics is the existence of interference. Given a sample space of sets of possible outcomes in a general process, classical probabilities obey the sum rule: $p(\mathcal{A} \cup \mathcal{B}) = p(\mathcal{A}) + p(\mathcal{B})$ for sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cap \mathcal{B} = \emptyset$. However, in standard quantum mechanics, this sum rule is not obeyed generally due to non-linearity of the amplitude. [10] An example is the double slit experiment: suppose there is no measurement on the position of the particle in either of the two slits, the probability of the particle found at a position z behind the slit is in general different from the sum of probabilities of the same result with only one slit open. This originates from the non-linearity of the probability in standard quantum mechanics and fact that the elementary amplitudes in standard quantum mechanics describe only one measurement.

A more subtle disadvantage of standard quantum mechanics is that only measurement result at a single precise time can be inferred. This, together with the effect of interference, makes computation of sequence of measurements difficult in general. This also imposes restrictions on the applicability of standard quantum mechanics on problems involving extends of time. These problems calls for a new treatment of quantum mechanics that has a wider applicability but also reproduces the measurement results under resemblance of measurement situations.

The idea of considering histories as a whole was not a new idea even before the first papers on consistent and decoherent approach histories. In 1948, Feynman reformulated non-relativistic quantum theory by expressing wavefunctions and transition amplitudes in terms of configuration space path integrals. [11] Feynman's path integral approach provides a new insight of quantum mechanics as a sum-over-history theory by considering all possible histories in the configuration space and assigning an appropriate weight to each history. The weight Feynman proposed was $e^{i\frac{S}{\hbar}}$ where S is the classical action along the associated history. This weight is the amplitude for the particular path being realised. To obtain the overall probability of spacetime transition, the square of the amplitude is needed so the interference between different spacetime histories still plays a significant role. Besides, while the spacetime path-integral approach can be applied to questions involving spacetime intervals, it is preferable to have a more general approach that can answer questions about general physical variables over a period of time.

The decoherent histories approach set out for a possible solution to get around the stated inconveniences while retaining the theoretical results of standard quantum mechanics. Without the implementation of any hidden variables [12], pilot waves [13] and any concepts of wave function collapse, the formulation removed the fundamental role of measurements. In such, the theory provides a possible way for an objective theory of quantum realm in which situations with high degree of interference are not considered. This allow a physical description of a closed system in quantum realm, such as the early universe.

The decoherent histories approach is a reformulation of quantum mechanics in terms of histories. It was first introduced by Griffiths [14] as the consistent histories (CH) approach in 1984 as an alternative to the standard quantum mechanics. Omnès [15–17]gave a more rigorous mathematical formulation for Griffith's idea in terms of logics. More precisely, Omnés reformulated Griffiths' idea in terms of general Boolean algebra and logical reasoning. Gell-Mann and Hartle [18–22]developed independently a closely related idea to the consistent histories approach, namely the decoherent histories approach, with a different motivation. Their objective is to look for a quantum theory that can describe closed systems and a framework which may be compatible with quantum cosmology, which is a problem for standard quantum mechanics.

In Section 2, mathematical formulations by Griffiths, Omnès, Gell-Mann and Hartle and Isham are summarised. The relevant mathematical concepts are also briefly discussed.

In DH approach, a system is completely defined by a set of histories that satisfies certain completeness, orthogonality conditions and a measure of interference between histories called the decoherence function. Physically, the completeness conditions requires at least one history to be a true description of the system and the orthogonality condition ensures that only one of the history is the true description of the system. Given a set of histories with zero interference with respect to the decoherence functional, the measures of histories in the set have properties consistent with those of classical probability, including the classical sum rule.

In standard interpretation, only decoherent sets of history are relevant in the DH approach. In a decoherent set of histories, the measure of each particular history are interpreted as the probability of that history being the true description of the system. Some consistent sets of histories can be combined into a larger set and still be decoherent. Such sets are said to be compatible with each other and the properties possessed by the two systems can be considered without contradictions. Incompatible sets generally contain non-commutative operators at some time points.

Closely related to the decoherence function is an operator called the C-representation. With this representation, the standard form of decoherence functional is in fact a bilinear functional on the sets of C-representations. However, it is not an bijection, which means given an operator in the Hilbert space, it can represent more than one history. C-representation also has other disadvantage such as its difficulty to construct under logica OR operation on two histories. Isham suggested another representation which he called the HPO theory of DH approach [23]. Isham also attempted to generalise the ideas HPO theory to be some abstract sets with some particular structure and properties. He also extended the mathematical framework to accommodate histories that describes events at different time points. These are summarised in Section 2.

As described in Section 2, any decoherent sets of histories can be equipped with a Boolean algebra to allow for logical reasoning to be made within the single set. For example, one can ask for the probability of "History 1 OR History 2", which means the probability of either history being true. It is also possible to make prediction such as the probability of a future description being true given a past history is true.

In Section 3, the possible interpretations of the DH approach and critiques on them are reviewed in detailed. Formulated as a quantum theory with no hidden variables, the realism feature of DH approach introduces paradoxes and controversies over interpretations of the quantum theory. The most common critiques on DH approach are on three aspects:

1. interpretations of the probabilities within a single consistent set

2. interpreting incompatible sets of histories at equal footings

3. emergence of quasi-classical realm

The original pioneers of the DH approaches [9, 24–26], together with d'Espagnat [27, 28], Kent and Dowker [29], Bassi and Ghirardi [30] devoted enormous amount of effort on the interpretations of the decoherent histories approach. The discussion on Section 1 and 2 lead to Grffiths' proposal of Logic [9] and the "single family rule" [14].

Situations in which systems decohere with each other, including correlations with quasiclassical domains and coupling with background environment, [31] were already investigated before the first papers on DH approach. Quasiclassical systems are extremely unstable in the DH approach. [15] However, our daily experience suggest the existence of at least one persisting quasiclassical realm. To account for this, Omnès [15,17], Gell-Mann and Hartle [18–22, 32]provided detailed explorations on emergence of quassiclassical systems and the psychophysical parallelism of a persisting in the setting of DH approach. These are discussed in detail in Section 3.

The applications of DH approach is discussed in Section 4. The intriguing properties of the DH approach such as the objective description of a system without the necessity of a measuring process and the ability to resemble quasiclassical properties, which are the most controversial issues in orthodox quantum mechanics, attracted many possible applications. Some applications includes describing EPR paradox [6, 8]without the existence of any action over a distance. Problems about the emergence of quasiclassical realm and in which a range of time has to be considered were also investigated. [33–37] Attempts on generalising the DH approach to quantum field theories has also been done. [38, 39]Such work and the results are discussed in Section 4.

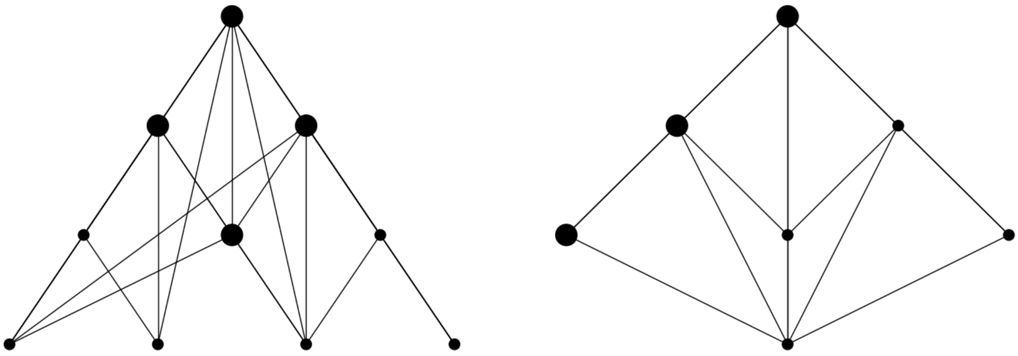
2 Mathematical Formalism

2.1 General Idea

Just as standard quantum mechanics can be formulated in terms of Hilbert space and path-integrals, the decoherence histories approach to quantum mechanics can also be formulated in terms of operators and path-integrals. The former was developed by Griffiths, Gell-Mann and Hartle, and the latter was elaborated in great extent by Hartle [40]. The operators method can describe a large class of histories but it relies on the existence of an external physical time. On the other hand,, the path integral formulation can describe histories even when such external clocks do not exist when generalised properly. Rather, the existence of time can be viewed as an emergent property of the system using path integral method. It is in this way that the path integral formulation is more useful when attacking problems of time in quantum mechanics and in developing a theory of quantum cosmology.

The two methods share some similarities but also contain differences. Both methods contain three fundamental elements: (1) a class of fine-grained histories (2) an operation called coarse graining on the fine-grained histories and subsequent histories; together with this the opposite operation called fine-graining (3) a decoherence functional on histories.

For a spacetime history, the decoherence functional in one method can be derived from another when well-behaving functionals of spacetime paths, including the paths themselves, are concerned. The two formulations are equivalent in such aspect. However, their structural properties differ greatly, e.g., the Hasse diagram under coarse and fine graining. (for example see Figure 1) This fundamental difference arises as the consequence of the fact that the set at each time point, there can only be one unique exhaustive and



(a) *(b)*

Figure 1: Examples of Hasse diagram representing the coarse graining in (1a) operator method; and (1b) path integral method. Each dot represents a family of histories in each formulation. The large dots represent decoherent families. The dots at the bottom are the completely fine grained families, the top dot represents the completely coarse grained family. Dots at higher level represents more coarse grained families. Dots connected by an edge represents are related by coarse and fine graining. The coarse graining of a decoherent family is also decoherent. The operator method allows more than one completely fine grained families, where as the path integral method only contain one unique completely fine grained family.

exclusive set of configuration space coordinates; whereas a system can be described by events chosen from one out of several complete and orthogonal sets of operators.

2.2 Operator Formulation

The basic quantities in the operator formulation are events. Events are the building block of histories in operator method.

In classical mechanics, an event is a statement concerning a system's state, e.g., "a tree is in a region M at some time t". Such a statement can either be true, false or meaningless. It is true if it is observed or is expected to be observed according to the deterministic and realistic classical equation of motion. It is false if its negation, for example the negation of the statement above is "a tree is not in a region M at some time t", is true. If it is neither true nor false, it is meaningless. For instance, a statement can be meaningless if the time is not specified.

The same definition of an event applies in quantum mechanics, i.e., an event in quantum mechanics is a statement such as "a system is in a state $|\Psi\rangle$ at some time t". At a given time t_k, a system in state $|\Psi\rangle$ is described by a projection operator $\hat{E} = |\Psi\rangle\langle\Psi|$. Therefore in consistent/ decoherent histories approach, an event is a projection operator at some time t_k. In order to be able to consider all possibilities for a given system, it is convenient to consider sets of events $\mathcal{A}_k := \{E_k^{\alpha_k};\ 1 \leq \alpha_k \leq \mu_k\}$ that sums up to the identity operator, i.e.,

$$\sum_{\alpha_k=1}^{\mu_k} \hat{E}_k^{\alpha_k} = \mathbb{I} \tag{2.1}$$

Such sets of events are said to be *exhaustive*. In order to simplify the problem more so that a given system can only be in the image of one projector, it is also convenient to add the restriction to sets of event that satisfies:

$$\hat{E}_k^{\alpha_k} \hat{E}_k^{\beta_k} = \delta_{\alpha_k \beta_k} \hat{E}_k^{\alpha_k} \tag{2.2}$$

Any sets of events that satisfies (2.2) are said to be *exclusive*. In standard consistent/ decoherence histories approach, only exhaustive and exclusive sets of events are considered. Given such sets of events, the corresponding *set of alternatives* $[\mathcal{A}_k]$ is the set consists in elements in $\mathcal{A}_k$ and all possible sums of two or more element from $\mathcal{A}_k$. For example, if $\mathcal{A}_k = \{\hat{E}_k^1, \hat{E}_k^2, \hat{E}_k^3\}$, then $[\mathcal{A}_k] = \{\hat{E}_k^1, \hat{E}_k^2, \hat{E}_k^3, \hat{E}_k^1 + \hat{E}_k^2, \hat{E}_k^2 + \hat{E}_k^3, \hat{E}_k^1 + \hat{E}_k^3, \hat{E}_k^1 + \hat{E}_k^2 + \hat{E}_k^3 = \hat{I}\}$. In general, for an exhaustive and exclusive set of events $\mathcal{A}_k$ with size μ_k, the corresponding $[\mathcal{A}_k]$ has $2^{\mu_k} - 1$ elements. Given the $\mathcal{A}_k$ consists in projection operators only, it is easy to show that $[\mathcal{A}_k]$ only contains projection operators.

Given a time order $t_1 < t_2 < \cdots < t_n$ and an exclusive and exhaustive set $\mathcal{A}_k$ of events at each time t_k, the associated *family of alternative histories* is the set $\mathcal{F}_{\{\mathcal{A}_k\}}$ of all possible sequences, called *histories*:

$$\mathcal{H}_{(\zeta)} := (\hat{G}_1^{\zeta_1}, \cdots, \hat{G}_n^{\zeta_n}) \tag{2.3}$$

where the subscript $\zeta = (\zeta_1, \cdots, \zeta_n)$ and $\hat{G}_k^{\zeta_k} \in [\mathcal{A}_k]$ for all k. The set of time points appearing in the history is called the *temporal support* of the history. Histories of the form as in (2.3), i.e., a sequence of projection operators, are said to be *homogeneous*.

Two homogeneous histories are *disjoint* if at least at one time t_k, the image of the corresponding projections are orthogonal.

A family of histories $\mathcal{F}_{\{\mathcal{B}_k\}}$ is a *coarse graining* of $\mathcal{F}_{\{\mathcal{A}_k\}}$ if $\mathcal{B}_k$ contains sums of projection operators in $\mathcal{A}_k$ for some t_k. $\mathcal{F}_{\{\mathcal{A}_k\}}$ is said to be a *fine graining* of $\mathcal{F}_{\{\mathcal{B}_k\}}$. Coarse graining forms a partial order on the set of families. The completely coarse grained family is the history with identity at all time points.

In Griffith's formulation, given a family of alternative histories with initial state $\hat{\rho}$, a weight is assigned to each histories $\mathcal{H}_{(\zeta)} := (\hat{G}_1^{\zeta_1}, \cdots, \hat{G}_n^{\zeta_n})$ by

$$p(\mathcal{H}_{(\zeta)}) = \mathrm{Tr}(\hat{G}_n^{\zeta_n}(t_n) \cdots \hat{G}_1^{\zeta_1}(t_1)\hat{\rho}(t_0)\hat{G}_1^{\zeta_1}(t_1) \cdots \hat{G}_n^{\zeta_n}(t_n)) \tag{2.4}$$

where the operators in the trace are in Heisenberg picture. For a pure initial state $|\Psi\rangle\langle\Psi|$, Eq (2.4) is equivalent to the norm of $\hat{G}_n^{\zeta_n}(t_n) \cdots \hat{G}_1^{\zeta_1}(t_1)|\Psi\rangle$ so the weight is non-negative. In a mixed state, the initial state is a positive combination of pure state density operators, therefore in any cases, given a homogeneous history $\mathcal{H}_{(\zeta)}$,

$$p(\mathcal{H}_{(\zeta)}) \geq 0 \tag{2.5}$$

A family of alternative histories is said to be *consistent* if for all k, given $\hat{G}_k^{\zeta_k} = \sum\limits_{\mu_k \in \mathcal{I}} \hat{E}_k^{\mu_k}$:

$$p(\hat{G}_1^{\zeta_1}, \cdots, \hat{G}_k^{\zeta_k}, \cdots, \hat{G}_n^{\zeta_n}) = \sum_{\mu_k \in \mathcal{I}} p(\hat{G}_1^{\zeta_1}, \cdots, \hat{E}_k^{\mu_k}, \cdots, \hat{G}_n^{\zeta_n}) \tag{2.6}$$

A single history is consistent if the smallest family containing it is consistent. It can be shown that the consistency condition (2.6) is equivalent to

$$\mathrm{Re}(\mathrm{Tr}(\hat{G}_n^{\zeta_n'}(t_n)\cdots\hat{G}_1^{\zeta_1'}(t_1)\hat{\rho}(t_0)\hat{G}_1^{\zeta_1}(t_1)\cdots\hat{G}_n^{\zeta_n}(t_n))) = 0 \ \ \forall\zeta',\zeta \tag{2.7}$$

Given a family of histories, the *decoherence functional* of two homogeneous histories is defined as:

$$D(\mathcal{H}_{(\zeta')},\mathcal{H}_{(\zeta)}) = \mathrm{Tr}(\hat{G}_n^{\zeta_n'}(t_n)\cdots\hat{G}_1^{\zeta_1'}(t_1)\hat{\rho}(t_0)\hat{G}_1^{\zeta_1}(t_1)\cdots\hat{G}_n^{\zeta_n}(t_n)) \tag{2.8}$$

Two homogeneous histories *decohere* when

$$D(\mathcal{H}_{(\zeta')},\mathcal{H}_{(\zeta)}) = 0 \tag{2.9}$$

So the diagonal of the decoherence functional contains the weight of individual histories in the family. A family of histories is decoherent if every pair of homogeneous histories decohere. Two decoherent families of histories are *compatible* if they can be combined to form another decoherent family of histories.

Given a decoherence functional D, for any given pair of histories $\mathcal{H}_{(\zeta)}$ and $\mathcal{H}_{(\zeta')}$, it can be shown that the decoherence functional satisfies an analogue of Cauchy–Schwarz inequality: [41]

$$|D(\mathcal{H}_{(\zeta)},\mathcal{H}_{(\zeta')})|^2 \leq D(\mathcal{H}_{(\zeta)},\mathcal{H}_{(\zeta)})D(\mathcal{H}_{(\zeta')},\mathcal{H}_{(\zeta')}) \tag{2.10}$$

Condition (2.10) is particularly useful in approximations in the computation of decoherence of histories describing hydrodynamical variables.

As stated earlier, the weight defined by Eq (2.4) satisfies positivity. When a family of histories satisfies the condition 2.9 and hence the consistency condition 2.7, this weight satisfies the classical sum rule. As a consequence of this and exhaustive property of the events at each single time, when the weights of all histories summed to $\mathrm{Tr}(\hat{\rho}) = 1$, i.e., unity. Therefore the weight in a decohering family of history can be viewed as a probability for each history. This is the standard interpretation of the weight in CH/ DH approach. [14, 18–21]

Given a family $\mathcal{F}_{\{\mathcal{A}_k\}}$ of histories, a Boolean algebra can be associated so that logical statements, e.g., predictions, can be made about a system. It can be constructed as suggested by Omnès [15]:

> A projection operator $\hat{G}_k^{\zeta_k}$ on a given Hilbert space can be associated with its image $\eta_k^{\zeta_k}$. So each histories $\mathcal{H}_{(\zeta)}$ can be associated with a subspace $\xi_{(\zeta)} = \eta_1^{\zeta_1} \times \eta_2^{\zeta_2} \times \cdots \eta_n^{\zeta_n}$ in the Hilbert space. This can be repeated for all histories to obtain a set $\mathfrak{F}$. The space $\mathfrak{B}$ of all possible unions in $\mathfrak{F}$ forms a Boolean algebra by
>
> $$\xi_{(\zeta)} \vee \xi_{(\zeta')} = \xi_{(\zeta)} \cup \xi_{(\zeta')}, \quad \xi_{(\zeta)} \wedge \xi_{(\zeta')} = \xi_{(\zeta)} \cap \xi_{(\zeta')}, \quad \neg\xi_{(\zeta)} = \mathfrak{B} \setminus \xi_{(\zeta)}$$
>
> The Boolean algebra $\mathfrak{B}$ defines Boolean operations on the set of histories. $\mathcal{H}_{(\xi)} \vee \mathcal{H}_{(\xi')}$ is the history associated with $\xi_{(\zeta)} \vee \xi_{(\zeta')}$, $\mathcal{H}_{(\xi)} \wedge \mathcal{H}_{(\xi')}$ is the history associated with $\xi_{(\zeta)} \wedge \xi_{(\zeta')}$. $\neg\mathcal{H}_{(\xi)}$ is the history associated with $\mathfrak{B} \setminus \xi_{(\zeta)}$.

Given a family of histories, and that $p(\mathcal{H}_{(\zeta')}) \neq 0$, the conditional weight of $\mathcal{H}_{(\zeta)}$ given $\mathcal{H}_{(\zeta')}$ is defined as

$$p(\mathcal{H}_{(\zeta)}|\mathcal{H}_{(\zeta')}) = \frac{p(\mathcal{H}_{(\zeta)} \wedge \mathcal{H}_{(\zeta')})}{p(\mathcal{H}_{(\zeta')})} \tag{2.11}$$

When the family is decoherent, the conditional weight becomes the conditional probability.

A general history is not necessarily homogeneous. A history formed under $\wedge$ can be written as a sequence of projections such that at each time the operator projects onto the intersection of the projections in the original histories. Therefore, such histories are homogeneous. However, similar definition does not work for $\vee$. For example, ("an electron is at x_1 at t_1 then at x_2 at t_2" or "the electron is at y_1 at t_1 then at y_2 at t_2") is not the same as ("an electron is at x_1 or y_1 at t_1 then at x_2 or y_2 at t_2"). Therefore, the statement upon $\vee$ is in general a sum of homogeneous histories. Since the negation of a statement in general involves "or", e.g., the negation of "the electron is at x_1 at t_1" is the statement ("the electron is at x_2 at t_1" or "the electron is at x_3 at t_1 or $\cdots$), negation of a homogeneous history is also inhomogeneous in general.

Eq (2.7) and (2.9) implies that the only factor that affects consistency or decoherence of two homogeneous histories are the products $\hat{C}[\mathcal{H}_{(\zeta)}] = \hat{G}_1^{\zeta_n}(t_1)\cdots\hat{G}_1^{\zeta_1}(t_1)$ and $\hat{C}[\mathcal{H}_{(\zeta')}]$. These are called the C-representation of histories. In terms of the C-representations, the decoherence functional between two homogeneous histories can be rewritten as:

$$D(\mathcal{H}_{(\zeta')}, \mathcal{H}_{(\zeta)}) = \mathrm{Tr}(\hat{C}[\mathcal{H}_{(\zeta')}]\hat{\rho}(t_0)\hat{C}[\mathcal{H}_{(\zeta)}]^\dagger) \tag{2.12}$$

While the C-representation is a convenient and straightforward way to represent homogeneous histories, it has some disadvantages. (1) the representation is not one-to-one so some information about the history is lost in the representation; (2) the representation is in general not a projection operator unless all the operators in the history commute. A projection representation is preferred because the representations upon logical operations, i.e., representations of inhomogeneous histories, can be computed in a natural way shown in Eq (2.14a), (2.14b), (2.14c).

These can be get around by using the HPO representation suggested by Isham. [23] In Isham's formulation, a homogeneous history $\mathcal{H}_{(\zeta)} := (\hat{G}_1^{\zeta_1}, \cdots, \hat{G}_n^{\zeta_n})$ is represented as

$$\Gamma(\mathcal{H}_{(\zeta)}) = \hat{G}_1^{\zeta_1} \otimes \cdots \otimes \hat{G}_n^{\zeta_n} \tag{2.13}$$

It is easy to see that this is one-to-one. For homogeneous histories, it is also a projection on the tensor product space $\otimes_{i=1}^{n}\mathcal{S}_i$, where $\mathcal{S}_i$ is the Hilbert space at time t_i. This allows a simple computation for the representation of inhomogeneous histories:

$$\Gamma(\mathcal{H}_{(\zeta)} \wedge \mathcal{H}_{(\zeta')}) = \Gamma(\mathcal{H}_{(\zeta)})\Gamma(\mathcal{H}_{(\zeta')}) \tag{2.14a}$$

$$\Gamma(\mathcal{H}_{(\zeta)} \vee \mathcal{H}_{(\zeta')}) = \Gamma(\mathcal{H}_{(\zeta)}) + \Gamma(\mathcal{H}_{(\zeta')}) - \Gamma(\mathcal{H}_{(\zeta)} \wedge \mathcal{H}_{(\zeta')}) \tag{2.14b}$$

$$\Gamma(\neg\mathcal{H}_{(\zeta)}) = \mathbb{I} - \Gamma(\mathcal{H}_{(\zeta)}) \tag{2.14c}$$

Eq (2.14c) is in fact a consequence of Eq (2.14a) and Eq (2.14b) by putting $\mathcal{H}_{(\zeta')} = \neg\mathcal{H}_{(\zeta)}$.

The C-representation of homogeneous histories can be recovered from the HPO representation by the function:

$$\Phi[\Gamma(\mathcal{H}_{(\zeta)})] = \hat{G}_n^{\zeta_n}(t_n)\cdots\hat{G}_1^{\zeta_1}(t_1) \tag{2.15}$$

The C-representations for inhomogeneous histories can be computed by linear extension of Φ. Then the weight of $\neg\mathcal{H}_{(\zeta)}$ is

$$\begin{aligned}
&\mathrm{Tr}((\mathbb{I} - C(\mathcal{H}_{(\zeta)}))\hat{\rho}(t_o)(\mathbb{I} - C(\mathcal{H}_{(\zeta)}))^\dagger) \\
&= 1 - \mathrm{Tr}(\hat{\rho}(t_o)C(\mathcal{H}_{(\zeta)})^\dagger) - \mathrm{Tr}(C(\mathcal{H}_{(\zeta)})\hat{\rho}(t_o)) + \mathrm{Tr}(C(\mathcal{H}_{(\zeta)})\hat{\rho}(t_o)C(\mathcal{H}_{(\zeta)})^\dagger)
\end{aligned}$$

For a single-event history, this returns the result $p(\neg\hat{G}^\zeta) = 1 - p(\hat{G}^\zeta)$, consistent with the probability of a measurement result in standard quantum mechanics.

Isham also proposed a general structure for the space of all histories $\mathfrak{K}$ and the space of homogeneous histories $\mathfrak{H}$. The space of homogeneous histories forms a partial semigroup with an operation $\circ$ of "precedes", together with a partial order $<$ known as "coarse graining" and a meet semilattice consistent with the partial semigroup. To $\mathfrak{H}$, there is a homomorphic partial semigroup $\mathcal{J}$. The elements of $\mathcal{J}$ are called the temporal supports. The space $\mathfrak{H}$ is embedded in a larger space $\mathfrak{K}$ with ortho-complemented lattice structure consistent with the structure of $\circ$. $\mathfrak{K}$ contains elements result from conjunction and negations of homogeneous histories.

A general decoherence functional is a complex, hermitian and positive function on the space of all histories. For diagonal elements to be able to be interpreted as probabilities when decoherence condition is satisfied, it is also required for the decoherence functional to be normalised. This is automatically satisfied in the operator method, but has to be examined more closely when arbitrary general histories are concerned.

Given a family of histories, it is said to satisfy the *strong decoherence* condition if for all histories, $\mathcal{H}_{(\zeta)}$ and their C-representations $C(\mathcal{H}_{(\zeta)})$, there is an operator $\mathcal{R}_{(\zeta}$) such that

$$\hat{C}(\mathcal{H}_{(\zeta)})\hat{\rho} = \mathcal{R}_{(\zeta)}\hat{\rho} \tag{2.16a}$$

$$\mathcal{R}_{(\zeta)}\mathcal{R}_{(\zeta')} = \delta_{(\zeta\zeta')}\mathcal{R}_{(\zeta)} \tag{2.16b}$$

$$\sum_{\zeta}\mathcal{R}_{(\zeta)} = \mathbb{I} \tag{2.16c}$$

A family is said to be *full* if it satisfies the strong decoherence condition with a unique and complete set $\{\mathcal{R}_{(\zeta)}\}$.

From (2.9) and the equivalent condition (2.7) of consistency, it is trivial that decoherence is a stronger condition than consistency. Moreover, it is showed that if a family saitifies the strong decoherence condition, it is decoherent. Therefore (2.9) is also called the *medium decoherence condition* and (2.7) is called the *weak decoherence condition.*

2.3 Path Integral Formulation

Similar to the operator method, the path integral formulation of decoherence histories approach consists of the completely fine-grained histories, an operation of coarse graining and fine graining, and the C-representation hence the decoherence functional. Given these three fundamental elements, probabilities can be assigned to histories as the diagonal elements of the decoherence functional when the off diagonal elements vanishes. Therefore, the only difference of the path integral formulation from the operator method can be is how the C-representations are defined. Once it is defined, the other parts of the formulation follows as those in the operator methods.

For configuration space paths, the fully fine grained histories are single paths. The C-representation of a path $\mathcal{Q}(t)$ can be expressed in terms of:

$$\langle q_f|\hat{C}[\mathcal{Q}]|q_i\rangle = e^{\frac{i}{\hbar}S[\mathcal{Q}]} \tag{2.17}$$

where $S[\mathcal{Q}]$ is the classical action function along the path $\mathcal{Q}(t)$. With the C-representation in hand, the decoherence functional for two fully fine-grained histories $\mathcal{Q}$ and $\mathcal{Q}'$ is given by Eq.2.12, which, in configuration path integral form, is equivalent to:

$$D(\mathcal{Q}', \mathcal{Q}) = \delta(q'_f - q_f)e^{\frac{i}{\hbar}(S[\mathcal{Q}']-S[\mathcal{Q}])}\langle q'_i|\hat{\rho}|q_i\rangle \tag{2.18}$$

Coarse graining of configuration space histories can be achieved in the same way as in the operator method, i.e., by considering a finite region ζ_t at each time. The C-representation of coarse grained histories can be obtained by functional integrating the C-representations over all paths passing through the regions ζ_t:

$$\langle q_f|\hat{C}[\mathcal{Q}]|q_i\rangle = \int \mathcal{D}\mathcal{Q}\; e^{\frac{i}{\hbar}S[\mathcal{Q}]} \tag{2.19}$$

and the decoherence functional is:

$$D(\mathcal{Q}', \mathcal{Q}) = \int \mathcal{D}\mathcal{Q} \int \mathcal{D}\mathcal{Q}'\ \delta(q'_f - q_f) e^{\frac{i}{\hbar}(S[\mathcal{Q}'] - S[\mathcal{Q}])} \langle q'_i | \hat{\rho} | q_i \rangle \tag{2.20}$$

This formulation can be easily generalised to a formulation in which coarse graining can be achieved by partitioning the whole configuration space into regions. The coarse graining on surfaces of constant time is only a special case of such generalisation. Such formulation can be applied to any theory which has a configuration space description, for example, space coordinates and general fields as in ref [40]. However, the coarse graining in the way stated above depends on the existence of a physical clock due to a fixed background spacetime geometry so that one can define surfaces of constant time. The theory, therefore, cannot be generalised to situations in which a well-defined background time cannot be assumed.

Nonetheless, Hartle suggested, in addition to coarse graining according to configuration space, coarse graining by partitioning paths according to some functionals of the paths. This way of coarse graining also includes coarse graining by configuration space: one can always define a functional of paths that returns the coordinates of the regions. Since this way of coarse graining takes the whole path instead of partition of a path into consideration, it can be expected to be useful when one needs to generalise quantum theory to situations without a well defined background spacetime.

Given a particular coarse graining η, the C-representation can be expressed in terms of sums over Eq.2.17:

$$\langle Q_f | \hat{C} | Q_i \rangle = \sum_{\mathcal{Q} \in \eta} e^{\frac{i}{\hbar} S[\mathcal{Q}]} \tag{2.21}$$

where the Q_f and Q_i are the final and initial values of the system. With an appropriate inner product $\cdot$ of state vectors, one can also define C-representation in terms of exhaustive and exclusive final and initial wavefunctions $\{\Psi_{b,f}(Q_f)\}$ and $\{\Psi_{a,i}(Q_i)\}$ with probabilities $\{p_{b,f}\}$ and $\{p_{a,i}\}$ as:

$$\langle \Psi_{b,f}|\hat{C}|\Psi_{a,i}\rangle = \Psi_{b,f}(Q_f)\cdot\langle Q_f|\hat{C}|Q_i\rangle\cdot\Psi_{a,i}(Q_i) \tag{2.22}$$

With Eq.2.22, one can define the decoherence functional for a general coarse graining of path as

$$D(\mathcal{Q}',\mathcal{Q}) = \mathcal{A}^{-1}\sum_{a,b} p_{b,j}\ \langle \Psi_{b,f}|\hat{C}'|\Psi_{a,i}\rangle\ \langle \Psi_{b,f}|\hat{C}|\Psi_{a,i}\rangle^*\ p_{a,i} \tag{2.23}$$

where $\mathcal{A}$ is a normalisation factor so that the sum of decoherence functionals over all paths yields unity.

Eq.(2.23) is equivalent to Eq.(2.12) in quantum mechanics described by Hilbert spaces in flat spacetime where the initial and final states are described by the density matrices:

$$\hat{\rho}_i = \sum_a p_{a,i}|\Psi\rangle_{a,i}\langle\Psi| \tag{2.24a}$$

$$\hat{\rho}_f = \sum_b p_{a,f}|\Psi\rangle_{b,f}\langle\Psi| \tag{2.24b}$$

This justifies the definition of decoherence functional Eq.(2.23).

3 Interpretation

3.1 Decoherent Family

3.1a. Logical Conclusions

As stated in Section 2, each decoherent family of histories can be associated with a Boolean algebra and that given a history $\mathcal{H}_{(\xi')}$, the conditional probability $p(\mathcal{H}_{(\xi)}|\mathcal{H}_{(\xi')})$ of $\mathcal{H}_{(\xi)}$ is given by Eq (2.11). It was shown that this conditional probability, when it equals

1, satisfies the properties of logical implications (if...then...). [15] It is also conventional [14, 29]to interpret $p(\mathcal{H}_{(\xi)}|\mathcal{H}_{(\xi')}) = 1$ as $\mathcal{H}_{(\xi')} \Rightarrow \mathcal{H}_{(\xi)}$. In other words, if it is agreed that $\mathcal{H}_{(\xi')}$ happens, then it is certain that $\mathcal{H}_{(\xi)}$ happens. This interpretation is particularly important in the investigation of interpretations involving incompatible families.

3.1b. Probabilities of Histories

The similarity between the standard quantum mechanics and the decoherence histories approach is that both of them are not meant to be a hidden-variable theory. Both approaches require a probabilistic interpretation, the only property that differentiates one history from another within the same family in the approach are the probabilities of different histories. However, whereas the probabilities in standard quantum mechanics refer to the distribution of measurements on certain observables under large number of repetition, the meaning of "probability of a history" is not clear.

As shown by Griffiths [14], when a measurement is compatible with a history, the conditional probability of the measurement result given the history coincides with the probability of standard quantum mechanics. Nonetheless, if the probabilities of decoherence histories approach can only be inferred when a measurement is performed, its interpretation can be none more than Copenhagen interpretation. In order to be a valid quantum theory of closed systems or a theory of quantum cosmology, the probabilities in a decoherent family have to be associated with some objective properties processed by the system. A simple analogy is the probabilities of "head"/"tail" after throwing a coin illustrates the physical properties processed by the coin.

The analogy between the probability of classical statistical mechanics and decoherent histories approach was extended further to the possibility to assign intrinsic values to histories in a way that it is meaningful to ask if a particular molecule in a gas has a specific energy even though it is given by a probability. The abstract mathematical statement is that given a sample space and a probability distribution over the elements of the sample space, a homomorphism from the sample space to $\{0, 1\}$ can be constructed. This homomorphism is the *truth value* [26, 30]and it is natural to assume this, in the context of decoherent histories approach, refers to whether the particular history correctly describe the system or not.

This interpretation works well within a single decoherent family. However, as elaborated by Bassi and Ghirardi [30] and explained in the next part, when the idea is extended to incompatible families, contradictory conclusions can result without appropriate auxiliary rules.

3.1c. Causality Paradox

The paradox concerning causality in decoherence history approach was first raised by d'Espagnat [27]. His argument was to consider any consistent three-times history:

$$G_0 \to G_1 \to G_2 \tag{3.1}$$

In his article, d'Espagnat considered the spin measurement of a spin-1/2 particle. G_0 is the event "$m_z = \frac{1}{2}$ at t_0", G_1 is the event "$m_z = \frac{1}{2}$ at t_1", G_2 is the event "$m_x = \frac{1}{2}$ at t_2".

He argued that if this history is the true history describing the system in question, so is part of the history:

$$G_0 \to G_1 \tag{3.2}$$

This partial history alone can be considered as a history alone. Causality demands that changing what happens at t_2 cannot change what happens at earlier times. In particular, what happens at t_2 cannot affect the physicality of the history (3.2). On the other hand, it is possible to find an event G_2', e.g., $m_y = \frac{1}{2}$ in d'Espagnat's example, at t_2 such that the history

$$G_0 \to G_1 \to G_2' \tag{3.3}$$

is inconsistent. So it seems that changing G_2 to G_2' at t_2 changes the physicality of previous history, violating causality.

However, it is important to understand that in decoherence history approach, physicality of a history, hence any partial histories in it, depends on all events of the history. This can be observed from Eq (2.8) and Eq (2.9). So another way to see the causality paradox is that the what happened in earlier times in the restricts which events are valid in the later times. In the example of (3.1) and (3.3), G_2 is a valid "future" event whereas G_2' is not.

An interesting scenario was shown in Griffiths' reply to d'Espagnat [9]. His example is to consider the history of a photon in an interferometer with two beam splitters, one, BS1, at the beginning of the interferometer, the other one, BS2, at the end of the interferometer before two photon detectors. When BS2 is removed, the consistent history considered has a final state at either of the photon detector. When BS2 is inserted, holding to the same partial history before reaching BS2 as before, the final state has to be a superposition state. In other words, decoherent histories approach without causality violations allows physical interpretations of superposition states.

This, nonetheless, is not as frustrating as it seems. A superposition state is none other than a quantum state, which is a valid state in a quantum theory. In standard quantum mechanics, superposition states are confusing because a system is completely described by a state at a particular time. On the other hand, in the decoherence histories approach, the complete description of a system is given by the entire history. The problem concerning superposition state does not arise in decoherence histories approach because now the superposition state is simply a part of the history.

3.1d. Approximate Decoherence

In practice, one often encounter cases in which histories are not exactly decoherent but in certain limit the deviation from exact decoherence is sufficiently small that is physically undetectable. This is common and useful especially in investigating emergence of macroscopic phenomenon from quantum theory. [31, 33–37, 42, 43]. In general, approximate decoherence arises when microscopic and macroscopic systems coexists so that quantum effects are non-vanishing yet too small to be significant. Some physicists such as Griffiths [14], Gell-Mann and Hartle [21] accepts the existence of such small deviations from exact decoherence so long as physical detections and interpretations are concerned.

On the other hand, such views received criticisms. Dowker and Kent argued that such approximations should not be given a fundamental role in a precise theory:

> "More fundamentally, many physicists would prefer a theory in which probabilities are precisely defined and precisely obey the sum rules. The consistent histories formalism does, after all, have a natural mathematical structure: why weaken it with ad hoc prescriptions?" [29]

Analogous to perturbation theory, one only requires the expansion up to certain order but the theory behind is exact, either in the form of a differential equation or a matrix equation. Another example is the use of Newtonian formula in low-energy classical mechanics. One can loosen the exactness in computation when the approximation does not affect the result in a detectable way but the theory itself needs to be exact.

Dowker and Kent also pointed out other problems caused by accepting approximately decoherent families into the theory such as classifications of families [32] because this hugely increases the number of families one needs to consider. They also argued that given an appropriate measure of distances between families, one can always find a consistent family near an approximately consistent one. This idea was made more precise and criterions under which such families exist were explored in depth by Dowker and Halliwell [41] and McElwaine [44]. The crucial point of this is that if one finds an approximately decoherent set, an exact decoherent one can be associated so there is no necessity to use approximately decoherent families.

Dowker and Halliwell also found a criterion for approximate decoherence to order ϵ:

$$D(\mathcal{H}_\xi, \mathcal{H}_{\xi'})D(\mathcal{H}_{\xi'}, \mathcal{H}_\xi) \leq \epsilon^2 \, D(\mathcal{H}_\xi, \mathcal{H}_\xi)D(\mathcal{H}_{\xi'}, \mathcal{H}_{\xi'}) \tag{3.4}$$

which becomes vital in analysing approximate decoherence.

3.2 Incompatible Sets

3.2a. Contradictory Conclusions

It was already noticed by Griffiths in his first article on consistent histories approach [14] that combining logical conclusions from incompatible consistent families of histories can lead to problematic results. Kent [45] also showed that interpreting incompatible families together leads to contradictory retrodictions and predictions, for instance, one can end up with a situation in which one event implies orthogonal events.

Another contradiction due to interpreting incompatible families together was analysed in detail by Bassi and Ghirardi [30]. They tried to extend the assignment of truth values discussed in Section 3.1b. to any families that contain the same history. Since their arguments are useful in illustrating rules in decoherent histories approaches in the next parts, it is beneficial to express their ideas in this article.

Their arguments are based on an array concerning spins of a two-particle system considered by Mermin [46] and Peres [47]:

$$\begin{array}{ccc} \sigma_x^1 & \sigma_x^2 & \sigma_x^1\sigma_x^2 \\ \\ \sigma_y^2 & \sigma_y^1 & \sigma_y^1\sigma_y^2 \\ \\ \sigma_x^1\sigma_y^2 & \sigma_y^1\sigma_x^2 & \sigma_z^1\sigma_z^2 \end{array} \tag{3.5}$$

where $\sigma_i^1 = \sigma_i \otimes \mathbb{I}$, $\sigma_i^2 = \mathbb{I} \otimes \sigma_i$. It has the property that the operators in each single row and column commute. It can also be shown, e.g., by considering the eigenvalues of products along each row and column [25, 26], that there do not exist any consistent simultaneously assignment of values to each operators.

Consider 6 families of histories, each based on projections onto common eigenstates of operators along a particular row or column in of Mermin's magic square (3.5). The histories derived from the rows are:

$$(\sigma_{x+}^1\sigma_{x+}^2) \quad ((\sigma_{x+}^1\sigma_{x-}^2)) \quad (\sigma_{x-}^1\sigma_{x+}^2) \quad (\sigma_{x-}^1\sigma_{x-}^2) \quad ((\sigma_x^1\sigma_x^2)_+) \quad ((\sigma_x^1\sigma_x^2)_-) \tag{3.6a}$$

$$(\sigma_{y+}^1\sigma_{y+}^2) \quad (\sigma_{y+}^1\sigma_{y-}^2) \quad (\sigma_{y-}^1, \sigma_{y+}^2) \quad (\sigma_{y-}^1, \sigma_{y-}^2) \quad ((\sigma_y^1\sigma_y^2)_+) \quad ((\sigma_y^1\sigma_y^2)_-) \tag{3.6b}$$

$$((\sigma_x^1\sigma_y^2)_+) \quad ((\sigma_x^1\sigma_y^2)_-) \quad ((\sigma_y^1\sigma_x^2)_+) \quad ((\sigma_y^1\sigma_x^2)_-) \quad ((\sigma_z^1\sigma_z^2)_+) \quad ((\sigma_z^1\sigma_z^2)_-) \tag{3.6c}$$

The histories arising from the columns are

$$(\sigma_{x+}^1\sigma_{y+}^2) \quad (\sigma_{x+}^1\sigma_{y-}^2) \quad (\sigma_{x-}^1\sigma_{y+}^2) \quad (\sigma_{x-}^1\sigma_{y-}^2) \quad ((\sigma_x^1\sigma_y^2)_+) \quad ((\sigma_x^1\sigma_y^2)_-) \tag{3.7a}$$

$$(\sigma_{y+}^1\sigma_{x+}^2) \quad (\sigma_{y+}^1\sigma_{x-}^2) \quad (\sigma_{y-}^1\sigma_{x+}^2) \quad (\sigma_{y-}^1\sigma_{x-}^2) \quad ((\sigma_y^1\sigma_x^2)_+) \quad ((\sigma_y^1\sigma_x^2)_-) \tag{3.7b}$$

$$((\sigma_x^1\sigma_x^2)_+) \quad ((\sigma_x^1\sigma_x^2)_-) \quad ((\sigma_y^1\sigma_y^2)_+) \quad ((\sigma_y^1\sigma_y^2)_-) \quad ((\sigma_z^1\sigma_z^2)_+) \quad ((\sigma_z^1\sigma_z^2)_-) \tag{3.7c}$$

Observe that the first 4 histories in (3.6c) coincides with the last two histories in each of (3.7a) and (3.7b). Similarly, the first 4 histories in (3.7c) coincides with the last two histories in each of (3.6a) and (3.6b). In addition, it can be shown that [30] in the family (3.6c):

$$((\sigma_z^1\sigma_z^2)_+) = ((\sigma_x^1\sigma_y^2)_+ \wedge (\sigma_y^1\sigma_x^2)_+) \vee ((\sigma_x^1\sigma_y^2)_- \wedge (\sigma_y^1\sigma_x^2)_-) \tag{3.8}$$

$$((\sigma_z^1\sigma_z^2)_-) = ((\sigma_x^1\sigma_y^2)_- \wedge (\sigma_y^1\sigma_x^2)_+) \vee ((\sigma_x^1\sigma_y^2)_+ \wedge (\sigma_y^1\sigma_x^2)_-) \tag{3.9}$$

Similarly in family (3.7c):

$$((\sigma_z^1\sigma_z^2)_+) = ((\sigma_x^1\sigma_x^2)_+ \wedge (\sigma_y^1\sigma_y^2)_-) \vee ((\sigma_x^1\sigma_x^2)_- \wedge (\sigma_y^1\sigma_y^2)_+) \tag{3.10}$$

$$((\sigma_z^1\sigma_z^2)_-) = ((\sigma_x^1\sigma_x^2)_+ \wedge (\sigma_y^1\sigma_y^2)_+) \vee ((\sigma_x^1\sigma_x^2)_- \wedge (\sigma_y^1\sigma_y^2)_-) \tag{3.11}$$

If we focus on only the family (3.6a), and assume that both spins in x-direction points toward direction "+". Then a truth value "1" can be assigned to the histories $(\sigma_{x+}^1\sigma_{x+}^2)$ and $((\sigma_x^1\sigma_x^2))_+$ while the other histories have truth value "0". In other words, the system possesses certain intrinsic properties described by the truth values:

$$\begin{array}{cccccc} (\sigma_{x+}^1\sigma_{x+}^2) & ((\sigma_{x+}^1\sigma_{x-}^2)) & (\sigma_{x-}^1\sigma_{x+}^2) & (\sigma_{x-}^1\sigma_{x-}^2) & ((\sigma_x^1\sigma_x^2)_+) & ((\sigma_x^1\sigma_x^2)_-) \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \end{array} \tag{3.12}$$

On the other hand, if we focus on the family (3.6b) instead and assume another truth value table:

$$\begin{array}{cccccc} (\sigma_{y+}^1\sigma_{y+}^2) & ((\sigma_{y+}^1\sigma_{y-}^2)) & (\sigma_{y-}^1\sigma_{y+}^2) & (\sigma_{y-}^1\sigma_{y-}^2) & ((\sigma_y^1\sigma_y^2)_+) & ((\sigma_y^1\sigma_y^2)_-) \\ \hline 0 & 1 & 0 & 0 & 0 & 1 \end{array} \tag{3.13}$$

Then (3.10), (3.11), (3.12) and (3.13) lead to a truth value table of family (3.7c)

$$\begin{array}{cccccc} ((\sigma_x^1\sigma_x^2)_+) & ((\sigma_x^1\sigma_x^2)_-) & ((\sigma_y^1\sigma_y^2)_+) & ((\sigma_y^1\sigma_y^2)_-) & ((\sigma_z^1\sigma_z^2)_+) & ((\sigma_z^1\sigma_z^2)_-) \\ \hline 1 & 0 & 0 & 1 & 1 & 0 \end{array} \tag{3.14}$$

Similar process on (3.7a) and (3.7b) lead to a truth value table of family (3.6c)

$$\begin{array}{cccccc} ((\sigma_x^1\sigma_y^2)_+) & ((\sigma_x^1\sigma_y^2)_-) & ((\sigma_y^1\sigma_x^2)_+) & ((\sigma_y^1\sigma_x^2)_-) & ((\sigma_z^1\sigma_z^2)_+) & ((\sigma_z^1\sigma_z^2)_-) \\ \hline 0 & 1 & 1 & 0 & 0 & 1 \end{array} \tag{3.15}$$

The virtue of using truth values is they represent intrinsic properties of the system. However, by comparing the truth value tables (3.14) and (3.15), the last two histories coincides with contrary truth values. So this lead to contradictory conclusion on the intrinsic properties of the system.

3.2b. Logic

The concept of Logic (with a big L to distinguish from ordinary classical logic) was suggested by Griffiths [9] as a plausible explanation of the logical problems concerning incompatible histories raised by d'Espagnat. [28] The concept of Logic is that a different set of rules than the classical ones are required in the descriptions of the quantum realm. Griffiths himself used the analogy of how the concept of time changed from Newtonian mechanics to Einstein's relativity:

> "...To demand that physicists must utilize only the logical tools previously available is no more reasonable than insisting that cosmologists must use Euclidean geometry! ... if Logic must deal with situations which Aristotle

> never dreamed of, it is not surprising if some modifications are needed in the notion of "true" ... when it is applied to the quantum domain. Just as the concept of time was changed in a fundamental way through the introduction of special relativity, it is not unreasonable to suppose that Logic needs a new concept ..." [9]

Accompanying the new concept of Logic are True and False, which are the counterpart of true and false in Logic. Griffiths required for consistency that within a single family, True (False) is equivalent to true (false). In the analogy with relativity, the choice between certain incompatible families is similar to choosing different coordinate system in relativity. It is meaningless to describe dynamics in the same system using two incompatible families but whenever physicists use the same family, they must agree on True statements:

> "... if we are concerned with whether S_z is positive or negative for a certain spin-1/2 particle at a certain time, we must choose to talk about S_z, and this precludes discussing S_x, for the same particle at the same time. But two physicists who accept the consistent history approach ... when discussing the same family, will reach the same conclusions about what is True, provided they start from the same premises ... " [9]

In this analogy, a problem of classifying families of histories is arises. In this case, it is expected to say that two incompatible families are different ways to describe the same system when both of them have to have the exactly identical probability distribution and the same number of histories. There are many ways this can be achieved, for instance, one can apply the same unitary transformation to operators at all times, or one can apply unitary operations that commutes with the Hamiltonian to some operators. In such way, the decoherent histories approach describes the universe as an equivalent set of families. There are many ways to define equivalent families other than those suggested above, e.g., one can define fine grainings as equivalent operations [23] and define families related by fine graining as equivalent universe. There are also ways of classifying families based on quasiclassical domains such as those discussed by Gell-Mann and Hartle. [32]

Logic can be resolution to the problem of contrary inferences in decoherence histories approach, but as emphasised by Dowker and Kent [29] such interpretation requires a subjective element of spectator's choice. This is undesired in a theory of closed system.

3.2c. Single-Family Rule

A rule closely related to the concept of Logic is the Single-Family (S.F.) rule, which is also proposed by Griffiths. [14] This is a simple solution to the interpretation problem when incompatible families are considered together. The Single-Family rule states that a system can only be described by a particular family of histories. This is closely related to the concept of Logic in the sense that Logical statements are only meaningful within the same family.

It is trivial that many contradictions regarding incompatible families are solved when discussions about a system is restricted within a single family. For instance, a system can only be described by a single family containing either m_z or m_x. The S.F. rule also provide a resolution to the apparent contradiction of the Mermin-magic-square paradox raised by Bassi and Ghirardi.

In their argument, assumptions on truth values regarding non-commutative operators are made in separate single families, e.g. (3.6a) and (3.6b). These values are then used to compute those of other operators across families. This, though, violates the Single-Family rule by asserting operators in different families share the same truth value. The virtue of the Single-Family rule is to treat incompatible families as distinct systems:

> " ... when incompatible frameworks (families) turn up in some quantum discussion, it is best to think of them as referring to two different systems, or to a single system at two different times ... " [25]

Assuming same truth values for common operators in different families have the same truth values is as illegal as assuming two uncorrelated particles always have the same spins.

Whether this view should be extended to compatible families can be ambiguous. For example, given two compatible decoherent families $\mathcal{F}_{\{\mathcal{A}\}}$ and $\mathcal{F}_{\{\mathcal{B}\}}$, they can be compile into one decoherent family $\mathcal{F}_{\{\mathcal{C}\}}$. Such situations arise when the events in $\mathcal{A}$ and $\mathcal{B}$ commute, including the scenario when one of them is a coarse graining of the other. These three families can be viewed as different choices of descriptions of the same system. Just as one is free to choose which of two commutative observables to talk about, or both, in standard quantum mechanics. Moreover, from formula (2.8) the probabilities of each histories only depend on its event. So the same history has the same probabilities in any decoherent families. Furthermore, according to Omnès [15], logical deductions are independent of families. So practically this is plausible. However, just as in the case of Logic, this adds a subjective element into the description. So this is not a valid choice for an objective quantum theory.

On the other hand, in view of a unified Single Family rule, it is natural to view any different families as descriptions of different systems. In this interpretation, the three families $\mathcal{F}_{\{\mathcal{A}\}}$, $\mathcal{F}_{\{\mathcal{B}\}}$ and $\mathcal{F}_{\{\mathcal{C}\}}$ are different systems. This interpretation has an advantage that a system can only be described by a unique family. Thence any subjective choices due to compatibility are removed and is more favourable for an objective quantum theory.

3.3 Quasiclassical Domains

As emphasized by Gell-Mann and Hartle, there is no perfectly classical realm but only quasi-classical realm due to quantum fluctuations:

> "Such histories cannot be exactly correlated in time according to classical laws because sometimes their classical evolution is disturbed by quantum events. There are no classical domains, only quasiclassical ones." [18]

A recent article by Halliwell provided a way that can be a meaningful measure of quantumness, which can in turn be considered as a measure of classicality. [48]

A successful quantum theory need to be able to explain the quasi-classical behaviour and the persistence of such realms as is observed, if one is to believe that the universe is governed by the same fundamental rule. So quasi-classical behaviour in decoherent histories approach has been one of the major focus on the theory. [15–22, 29, 32, 49]Emergence of quasi-classical realms are discussed in Section 4. This section focuses on the descriptions of quasi-classical domain in decoherent histories approach.

In order to discuss quasi-classical behaviours in the language of decoherent histories approach, it is favourable to consider maximally fine-grained decoherent families so as to include descriptions as detailed as possible, consistent with decoherence condition. Normally, certain extent of coarse-graining is required until quasi-classical domains emerges, e.g., one has to consider a range of positions rather than an exact point in space. Such coarse-grainings suppress the strength of quantum fluctuations on the history.

Logical statements are one of the most important constituent in a physical theory. It allows one to make predictions on the future and retrodictions on the past and hence to test the theory. Moreover, if one can pin-point events that actually happened, one can restrict discussions of histories containing these "actual facts". Quotation marks are put around "actual facts" because there is an ambiguity of what actually happened. [29] In brief, the problem suggested by Dowker and Kent is that one is unsure about descriptions as recorded by others. Memories can be deceiving, even written records can be false. This already causes problem on logical predictions and retrodictions. The difference between "what actually happened" and "what we think happened" can be much more problematic if one is to include themselves into the universe under consideration, since one can only be sure about the latter in the perfect case. What actually happened is never known in this sense unless we have a complementary theory of human experience.

Suppose there is a settlement in the questions raised in the previous paragraph, one can hope to make logical statements on histories with accordance to the definition of implications in Section 3.1a.. Transition from quantum logic to classical logic involves transition from a Hilbert-space description to a phase space description of a system. This can be achieved using Weyl's functions. [15, 50]The Weyl's functions allows one to associate operators and operations on wavefunctions to functions on phase space, embedding quantum descriptions in a classical framework. In the phase-space description, Boolean algebras are associated to cells in phase space. Under certain macroscopic conditions, deterministic behaviour can be recovered as one take the $\hbar \to 0$. [15]

In decoherent histories approach, everything along a history are described by a set of projections. So, a quasi-classical domain is expected to correspond to some projections. Therefore, time evolution and persistence of quasi-classical domains are inseparable with development of projections along histories. However, Dowker and Kent showed that quasi-classical domains are unstable in the sense that a quasi-classical future of a quasi-classical domain is not guaranteed.

In order to explain the persistence of quasi-classical realms as observed, Gell-Mann and Hartle introduced the idea of information gathering and utilising systems (IGUSes). IGUSes are described as:

> "A type of creature which is coupled by some form of sensory organs to its environment, able to model the local environment by some form of the logical processing, and able to act on the results of its computations" [29]

and described by Gell-Mann and Hartle as

> "... a complex adaptive system that has evolved to exploit the relative predictability of a quasiclassical domain ..." [19]

IGUSes are generally some coarse grained projections and are hopefully able to provide an explanation for our perception of a persisting quasi-classical realms in spite of the instability of a quasi-classical realm. Yet, it lacks the theory behind the psycho-physical parallelism, i.e., the correspondence of IGUSes and how their perception works in the framework of decoherence history approach. Situation can become even more complicated when one considers branching – the dependence of identity decomposition on past histories. To get a complete resolution of the persistence of the quasi-classical universe, a theory of the neuroscience in terms of quantum mechanics seems inevitable. This also requires a rigorous quantum mechanical definition of IGUSes. Such definition should allow one to make identification on projections that corresponds to either parts or the whole of the same IGUS, e.g., the same person.

3.4 Many History Interpretation

The many history interpretation was suggested by Dowker and Kent [29]. This interpretation requires a concept of fully fine grained decoherent families. A family $\mathcal{F}$ is said to be an *extension* of another family $\mathcal{F}'$ if the temporal support of $\mathcal{F}'$ is a subset of that of $\mathcal{F}$. It is *maximally extended* if it cannot be extended to be a decoherent family with more than one histories having non-zero probability. A family with more than one histories having non-zero probability is *fully fine grained* if it is maximally extended and maximally fine grained.

Requiring the existence of more than one histories with non-zero probability ensures the family is not trivial. The aim of requiring maximally extended and maximally fine grained is to include all possible physical histories one can possess. The many histories interpretation states that the complete description of the nature is a set $\mathcal{P}$ containing histories, each chosen from a distinct fully fine-grained family with probability equal to its probability insi de the family.

This interpretation assigns a fundamental role to the probabilities in terms of a selection process. The way that histories in different decoherent families are included in the description of the same nature is analogous to the scenario of Griffith's Logic (Section (3.1a.)). This interpretation, however, does not invoked a fundamental role to the observer: each fully fine grained families participates in the selection process.

It is important in the many histories interpretation that only exactly decoherent families are considered so that any statements about a history are exact, instead of approximate yet undetectable down to certain order as when approximately decoherent families are considered into the theory. In fact, the only reason that one is willing to consider approximate decoherence is to allow the existence of quasi-classical domains. The many history interpretation allows one to explain the quasi-classical experience by asserting that in each family, the IGUSes experience the chosen history. This solves the stability problem naturally.

3.5 Unknown Set Interpretation

Similar to the many history interpretation, the unknown set interpretation was suggested by Dowker and Kent [29]. This interpretation is very simple in structure. The universe is described by a single history chosen from one decoheret family according to the probabilities in family. However, one do not know which set is chosen and which history is realised. In terms of this interpretation:

> "If we are willing to assume an agreed list of historical events, we can pin down some of the past sets of projections in the realized history. These, however ... will not generally allow us to make useful unconditional predictions." [29]

Also, since only one history is realised, there is no concept of emergence of quasiclassical domains. Quasiclassical experience are more like an illusion in this interpretation:

> " ... we have the impression that the realized history has been quasiclassical so far, and that is all there is to be said. As a practical matter, we should no doubt assume that the realized history will be quasiclassical in the future. Conditioned on this assumption, we can make probabilistic predictions with the same scientific content as those of the Copenhagen interpretation." [29]

4 Applications

4.1 EPR Paradox

Many applications of the consistent histories approach in standard quantum mechanical problems, i.e., situations involving certain measurement apparatus have been considered. [14] The resulting computations showed the ability of the approach to supply a quantum theory consistent with realism. This can be most convincingly demonstrated using the EPR experiment. [6]

The EPR paradox was first suggested by Einstein, Podolsky and Rosen as an argument against standard quantum mechanics and led to the Copenhagen interpretation from the reply of Bohr [7]. Two particles A and B are sent into local interaction with each other, during which some of their properties are correlated. At t_0 after this interaction they move away from each other without further local interactions. One can think of the interaction as a decay of a nucleus, and the correlated properties be the momentum or the spins. At some time t_1 their separation is spacelike and one of the correlated property of particle A is measured by device D_A. At some $t_2 > t_1$, the same property of particle B is measured by device D_B. The whole process is summarised in Figure 2.

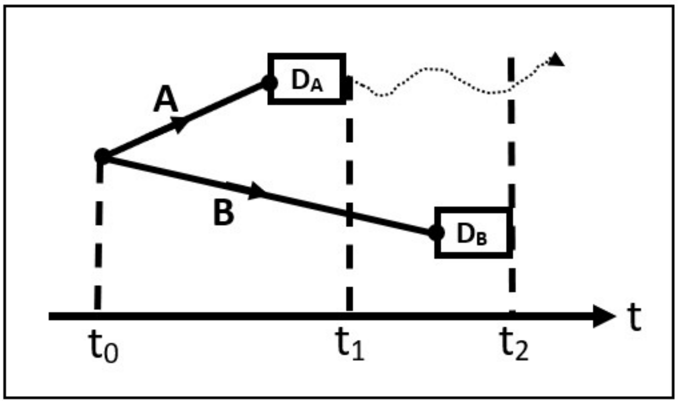


Figure 2: Diagram showing temporal sequence in EPR experiment.

In standard quantum mechanics, the eigenstates are tensor products of some single-particle eigenstates, consistent with the correlation. For instance, given the momentum must sum to zero, the possible two-particle eigenstates are of the form $|p\rangle \otimes |-p\rangle$. A general state of the two-particle system is a combination of such eigenstate. However, each particle can also be described by its own wavefunction. When a measurement is done on one particle, in standard mechanics, the wavefunction of that particle collapses. At the same time, due to correlation, standard quantum mechanics predicts that the wavefunction of the other particle has to collapse. However, the wavefunction collapse of the second particle is due to the collapse of the other one, which is separated by spacelike distance. There is an apparent "action over distance".

The difference between "action over distance" and our knowledge of the properties possessed by a distance system was made clear by Griffiths using an example of a 5g paper:

> "A sheet of paper weighing 5g is torn in two one is mailed to a scientist in London and the other to a scientist in Paris ... Weighing the piece in London has no physical effect upon the piece of paper in Paris. What it does affect is our knowledge about the latter, which is something very different." [8]

So the EPR problem in standard quantum mechanics is not the fact that one can refer the value carried by particle B upon measurement of particle A; rather the problem lies in the fact that the wavefunction of particle B immediately collapse to an eigenstate as a result of measurement on particle A. This violates the special relativity.

In general, one can consider the two-dimensional case in which D_A and D_B measure spin along different axis z_1 and z_2 so that the angle between them is θ. Denote $|\uparrow_1\rangle$ and $|\downarrow_1\rangle$ as spin up and down along z_1. Similarly, denote $|\uparrow_2\rangle$ and $|\downarrow_2\rangle$ as spin up and down along z_2. (See Figure 3)

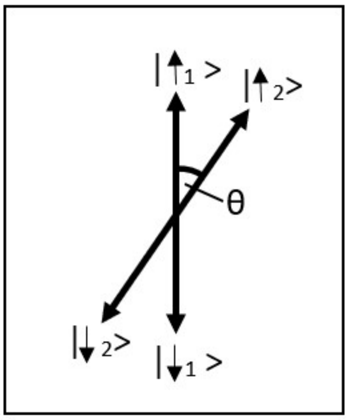


Figure 3: Axis along which measurements are made by z_1 and z_2.

The up and down states along z_1 and z_2 can be related by

$$|\uparrow_2\rangle = \cos\frac{\theta}{2}|\uparrow_1\rangle + \sin\frac{\theta}{2}|\downarrow_1\rangle \tag{4.1a}$$

$$|\downarrow_2\rangle = -\sin\frac{\theta}{2}|\uparrow_1\rangle + \cos\frac{\theta}{2}|\downarrow_1\rangle \tag{4.1b}$$

Denote $|\psi_{D_A}\psi_{D_B}\psi_A\psi_B\rangle$ as a general state of the system in which D_A is in state $|\psi_{D_A}\rangle$ and so on. Assume at t_0 the system is in the state

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|\psi_{D_A}\psi_{D_B}\uparrow_1\downarrow_1\rangle + |\psi_{D_A}\psi_{D_B}\downarrow_1\uparrow_1\rangle) \tag{4.2}$$

In other words, the spins of particles A and B anti-parallel. The state $|\Psi_0\rangle$ can be expanded in terms of $|\uparrow_2\rangle$ and $|\downarrow_2\rangle$ for particle B using Equations (4.1). At each t_k, consider the exhaustive and exclusive decomposition $\mathcal{A}_k$ of identity:

$$\begin{aligned}\mathcal{A}_1 = \{&|D_{A,\uparrow_1}\psi_{D_B}\uparrow_1\uparrow_2\rangle\langle D_{A,\uparrow_1}\psi_{D_B}\uparrow_1\uparrow_2|, |D_{A,\uparrow_1}\psi_{D_B}\uparrow_1\downarrow_2\rangle\langle D_{A,\uparrow_1}\psi_{D_B}\uparrow_1\downarrow_2|,\\ &|D_{A,\downarrow_1}\psi_{D_B}\downarrow_1\uparrow_2\rangle\langle D_{A,\downarrow_1}\psi_{D_B}\downarrow_1\uparrow_2|, |D_{A,\downarrow_1}\psi_{D_B}\downarrow_1\downarrow_2\rangle\langle D_{A,\downarrow_1}\psi_{D_B}\downarrow_1\downarrow_2|\}\end{aligned} \tag{4.3}$$

and

$$\mathcal{A}_2 = \{|D_{A,\uparrow_1} D_{B,\uparrow_2} \uparrow_1 \uparrow_2\rangle\langle D_{A,\uparrow_1} D_{B,\uparrow_2} \uparrow_1 \uparrow_2 |, |D_{A,\uparrow_1} D_{B,\downarrow_1} \uparrow_1 \downarrow_2\rangle\langle D_{A,\uparrow_1} D_{B,\downarrow_1} \uparrow_1 \downarrow_2 |,$$
$$|D_{A,\downarrow_1} D_{B,\uparrow_2} \downarrow_1 \uparrow_2\rangle\langle D_{A,\downarrow_1} D_{B,\uparrow_2} \downarrow_1 \uparrow_2 |, |D_{A,\downarrow_1} D_{B,\downarrow_2} \downarrow_1 \downarrow_2\rangle\langle D_{A,\downarrow_1} D_{B,\downarrow_2} \downarrow_1 \downarrow_2 |\} \tag{4.4}$$

and consider the family of histories:

$$\mathcal{F}: \ |\Psi_0\rangle \rightarrow [\mathcal{A}_1] \rightarrow [\mathcal{A}_2] \tag{4.5}$$

The family is decoherent so probabilities can be assigned to histories. The important results are [8]:

$$p \tag{4.6}$$

Equations ()-() show that the spins of particle A and B are correlated. Equations ()-() show that the spin measurement result of A and B are also correlated as expected. These, however, do not imply there is an action over distance between the particles and between the measuring devices. Griffiths showed that the probability distribution of spins of one particle is independent of another particle when they are not interacting. Therefore in decoherent histories approach, correlations do not require action over distance.

This result is an illustration of advantages of decoherent histories approach over the standard quantum mechanics:

1. the approach provides a local theory, i.e., there is no necessity of a action over distance; that is also able to
2. describe a closed system. In the discussion, measurement devices exist but they are part of the whole system. To be sure, one can consider another system in which there is no measurement device and obtain the same results ()-().

4.2 Hydrodynamic Variables

Hydrodynamic variables are essentially variables that describe flows in a system consisting of large amount of degrees of freedom. Such variables are important in quantum theory. One of the reasons is that they provide a connection between quantum realm and quasi-classical phenomenon. An example of this is the Ehrenfest theorem for position:

$$i\hbar\frac{d}{dt}\langle\hat{\mathcal{O}}\rangle = i\hbar\langle\frac{\partial}{\partial t}\hat{\mathcal{O}}\rangle + \langle[\hat{\mathcal{O}}, \hat{H}]\rangle \tag{4.7}$$

which, when considering the position operator, gives

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{m} \tag{4.8}$$

which establishes resembles the classical relation between momentum and position by considering the expectation value of the corresponding measurements. Putting Eq (4.8) into Eq (4.7) applied to momentum operator gives the ordinary Ehrenfest theorem:

$$m\frac{d^2}{dt^2}\langle x\rangle = -\langle\frac{\partial}{\partial x}V(x)\rangle \tag{4.9}$$

Given suitable initial states and Hamiltonians, e.g., a narrow Gaussian wavefunction evolving by a very slow smearing, Eq (4.9) gives a Newton law for the mean position of the wavefunction.

The decoherent histories analysis of hydrodynamic variables allows connections to be made between quantum realm and classical equations for more general hydrodynamic variables such as number densities, energy and spins. Furthermore, Ehrenfest theorem in standard quantum mechanic infer about the expectation values of measurements of the same variable over a large number of repetitions. Therefore this does not explain emergence of quasi-classical phenomenon. On the other hand, the decoherent histories approach attempts to analyse the problem by considering the histories of a variable, rather than the results of repeated measurements. This provide a more detailed descriptions of hydrodynamic variiables and thence the decoherent histories approach provides more insight into the emergence of quasi-classical phenomenon in a quantum theory fundamentally.

It was found that the histories of exactly conserved variables are exactly decoherent. [51] Therefore probabilities can be assigned to histories of the conserved variable. Classically, such quantities obeys a continuity equation which relates the evolution of the variables to the outward flow of it from the boundary of a region. Exactly conserved quantities emerges when one consider the limit that the volume to infinity. It is expected that the probabilities peak at those histories obeying the continuity equation and generally the hydrodynamic equations to which the conserved variable is associated.

If the histories of a hydrodynamic variable is approximately decoherent, one expects the histories to converge about those that obey the classical hydrodynamic equation. The decoherent histories analysis of hydrodynamic variables aims to (1) investigate the conditions that decohere the histories of hydrodynamical variables and (2) compute probabilities and look for situations that converges toward those that obey classical hydrodynamic equation. (2) is vital for treating quasi-classical realm as emergence phenomenon in quantum theory. (1) is critical for interpretation of quantum cosmology in which emergence of quasi-classical realms are important.

Halliwell and Brun investigated hydrodynamic variables in a weakly-interacting system, in which each component subsystem have the same initial state. [33] In their article, they confirmed that exact conservation of a hydrodynamic variable of the form

$$M_\Omega = \int_\Omega dV \rho(x) \tag{4.10}$$

the histories decohere and the only history with non-zero probability (hence $p(\mathcal{H}_\xi) = 1$) is the one along which the variable has the same value at each time point.

Halliwell showed that eigenstates of such hydrodynamic variables remains being approximate eigenstates given a large enough integration volume so that the effect of quantum fluctuations in the volume are neglectible. [35, 36, 52]Given this case, approximate decoherence can be achieved. The degree of decoherence of the whole system depends on the number and the degree of decoherence of component subsystems and the sizes of partition of the system. Given a system with sufficient large number of particles and a large partition, the adequate order of decoherence can be obtained.

Given approximate decoherence, the probabilities of histories peaks rapidly about the average values at each time. Given an initial state made up of a superposition of density eigenstates, the system is equivalent to a mixture of histories, each of which obey certain distinct hydrodynamic equations. As a result, if the initial state is in local equilibrium state, the whole system evolves around a single hydrodynamic equation. [53]

4.3 Arrival Time Problem

The *arrival time problem* concerns the problem of time in quantum mechanics. In standard quantum mechanics, measurements, represented as an operator, only refer to a single time point. In particular, the problem of time in quantum mechanics involves the fact that time is a parameter rather than an observable, i.e., an operator, in standard quantum mechanics. This problem also arises in quantum cosmology for a symmetric treatment of space and time since spacial position occurs in standard quantum mechanics as an operator. Attempts to solve the problem of time in quantum mechanics by defining a "Time operator" has been made. [54–56]

In practice, time is measured by macroscopic device, so coarse graining to a certain extent is required. Such situations is in contrary to the fundamental formulation of standard quantum mechanics. Feynman's sum-over-history approach is a possible way when the problem involves only spacetime regions and there have been attempts to apply this method to problems involving time in quantum mechanics. [57, 58]However, the problem becomes complicated very quickly when the problem involves non-trivial functions of space-time coordinates.

Decoherent histories approach, as a theory treating history fundamentally, provides a natural solution to the arrival time problem. The problem can be simplified as the probability that a particle to cross a spacial region within a certain time period. In order to apply decoherent histories approach, one, however, has to create the suitable decohering condition. Different methods, such as quantum Brownian motion model and hydrodynamical model, to induce decoherence in a closed system for analysing the problems concerning time of arrival in decoherent histories approach were suggested. [37]

The hydrodynamic model is to replicate the system N times where N is large so that results in hydrodynamic analysis can be applied. Such system is known to be decoherent and the computation is similar to standard techniques. [33] In the problem of arrival time, the relevant hydrodynamic variable is the number density. In this model, the problem is re-expressed as a hydrodynamic problem of number density crossing through a spatial region.

The quantum Brownian model decoheres the system by allowing the concerned system with other parts of the whole system. Specifically, the model submerges the concerned system into a thermal bath. Collision with foreign particles are not unlike frequent position measurements, which decoheres the spacetime histories of the particles. Therefore, in such model, it is natural to apply the spacetime path integral expression for the decoherence functional:

$$D(\mathcal{H}_x, \mathcal{H}_{x'}) = \int \mathcal{D}x \int \mathcal{D}x' \exp\left(\frac{i}{\hbar}(S[x] - S[x'] + W[x, x'])\right) \rho(x_0, x'_0) \tag{4.11}$$

where $W[x, x']$ induces dissipation and thermal fluctuations due to the thermal bath [59]. It is the existence of this term that is responsible for the decoherence of spacetime histories. In this model, it was shown that in consistent families, one obtain intuitive results, e.g., a particle in one side of a region divided into two parts stays at the same side if it keeps moving away from the margin between the two sides.

4.4 Quantum Fields and Quantum Cosmology

A decoherent histories approach to quantum field theory is vital for the approach to be a valid quantum theory. The reasons for this are that quantum field theory is (1) a quantum theory of fields (2) a relativistic quantum theory. In sharp contrast to non-relativistic quantum theory in which the fundamental subjects are localised particles, the fundamental subjects in the theory of quantum field are fields that spreads over space at each time. In terms of canonical quantisation, non-relativistic quantum mechanics requires [1]

$$[q_j, p_k] = i\hbar\, \delta_{jk} \tag{4.12}$$

On the other hand, canonical quantisation of boson fields, e.g., photon field, requires [60]

$$[\phi(x, t), \pi(x', t)] = i\hbar\, \delta(x - x') \tag{4.13}$$

Canonical quantisation of spinor fields such as electrons and higher spin fields involves more complications due to addition of spinor indices and replacement of commutator to anti-commutator for fermions. [61]

This fundamental difference of quantum fields with non-relativistic quantum mechanics suggests a fundamental change in the events in quantum field theory. It is expected that the histories in quantum fields theory concerns the value of the field at every spacetime point relevant. Therefore knowledge on the geometry of spacetime is very important related to a histories approach to quantum fields. In general, one has to consider quantum fields in curved spacetime as in the scheme proposed by Blencowe. [38] In his scheme, one considers a globally hyperbolic manifold $\mathcal{M}$ as a geometric description of spacetime. Each bounded open subset $\mathcal{J}$ of $\mathcal{M}$ is associated with a C^*-algebra $\mathcal{G}_{\mathcal{J}}$, members of which represent general fields. The class of self-adjoint elements in the C^*-algebra are identified with observables. Initial states in the framework are identified with the class of normalised, positive linear functionals on the C^*-algebra. A general maximally fine-grained history is of the form

$$\mathcal{H}_\xi : (G_1, G_2, \cdots, G_n) \tag{4.14}$$

where G_i describes the event "observable $\mathcal{O}_i$ has value α_i at spacetime point $\mathbf{x}_i$". This seems very similar to The decoherence functional of two histories:

$$\mathcal{H}_\xi : (G_1, G_2, \cdots, G_n) \tag{4.15}$$

$$\mathcal{H}_{\xi'} : (G'_1, G'_2, \cdots, G'_n) \tag{4.16}$$

with initial state ρ is:

$$d(\mathcal{H}_\xi, \mathcal{H}_{\xi'}) = \left(\prod_{i,j=1}^{n} \int d\lambda_i \int d\lambda'_j\right) e^{i \sum\limits_{i,j=1}^{n} -i(\lambda_i \mathbf{x}_i + \lambda'_j \mathbf{x}'_j)} \rho(e^{i\lambda_1 \mathcal{O}_1} \cdots e^{i\lambda_n \mathcal{O}_n} e^{i\lambda'_n \mathcal{O}'_n} \cdots e^{i\lambda'_1 \mathcal{O}_1}) \tag{4.17}$$

which is equivalent to Eq (2.8) when applied to standard histories in flat spacetime by identifying $\rho(e^{i\lambda_1 \mathcal{O}_1} \cdots e^{i\lambda_n \mathcal{O}_n} e^{i\lambda'_n \mathcal{O}'_n} \cdots e^{i\lambda'_1 \mathcal{O}_1})$ as the characteristic function, i.e, the Fourier transform, of Eq (2.8). Decoherence functionals on coarse-grained spacetime regions are obtained by integrating Eq (4.17) over the whole spacetime restricted to the regions relevant.

As mentioned, a decoherent histories approach to quantum field theory requires knowledge in the structure of spacetime. Therefore, the theory is inseparable with quantum cosmology. The dynamics of quantum cosmology can be obtained by solving a Wheeler-DeWitt equation: [62]

$$H\Psi_{universe} = 0 \tag{4.18}$$

where H is an operator containing dynamic information about the matter content and the gravitational field, $\Psi_{universe}$ is a funtion of the gravitational and matter field wavefunction of the universe, containing information of the gravitational and matter field in the universe. [63]

Wheeler-DeWitt equation suffers from a problem of time, e.g., the wavefunction of the universe cannot result in the emergence of a dynamic classical universe. [64–66]The problem of time in quantum cosmology is the result of the necessary existence of an "external clock" in standard quantum mechanics, whereas quantum cosmology is supposed to describe phenomenon including gravitational fields, hence time itself. This is a problem that the decoherent histories approach can get rid of.

Due to the ambiguity of time in quantum cosmology, one cannot genuinely ask about the probabilities involving time such as a particle entering a region of spacelike hyperspace within a certain time period. Nonetheless, one can always ask about the probability of the system following a certain orbit, or some arbitrary trajectories in a configuration space. The decoherent histories approach is naturally adapted to calculating probabilities of trajectories by using projections onto the trajectories. Halliwell published articles concerning probabilities of trajectories in simple models of quantum cosmology, for example a closed system governed by Klein-Gordon equation which has the same form as the Wheeler-DeWitt equation. [67–69]Such model is one of the simplest possible model for a quantum cosmological theory. Halliwell mainly used the path-integral approach, which is more convenient in computing amplitudes of trajectory, instead of the mathematical framework suggested by Blencowe.

5 Summary

Standard quantum mechanics developed by Schrödinger and Heisenberg, Jordan and Born are successful in reproducing the probability distribution of measurement results of a quantum theory. Possible measurement results are obtained through diagonalisation of the operator representing the corresponding observable. In other words, there are three elements in standard quantum mechanics:

1. there exists a functional taking two states as input and return a complex number.
2. this functional can be interpreted as the transition amplitude, the squared magnitude of which gives the probability, upon measurement.
3. diagonalisation of operators, the diagonal elements then represent the possible measurement results

Their formulation suffers from the two defects:

1. it cannot describe systems in which no distinguished macroscopic subsystem exists or acts as a measurement device. Closed systems are special cases of such type of systems because by definition no measurement can be performed.

2. it cannot describe situations in which no background clock exists, or in which time enters non-trivially.

Decoherent histories approach is able to get around these disadvantages by treating histories as a fundamental quantity. The fundamental elements in the decoherent histories approach are:

1. the class of completely fine grained histories

2. the process of coarse graining and its opposite, fine graining

3. a complex, positive, hermitian and normalised function taking in two histories and returning a number, called the decoherence functional

The decoherence functional in decoherence histories approach is the analogue of the amplitude functional in standard quantum mechanics. Both theories require a process of diagonalisation: operators in standard quantum mechanics; decoherence functional in decoherent histories approach. The fundamental role of measurement in standard quantum mechanics is replaced by the decoherence functional in decoherent histories approach: whereas probabilities arises in standard quantum mechanics as a consequence of measurement, it arises in decoherent histories approach when decoherence condition is satisfied.

Despite being a quantum theory of closed system, the decoherent histories approach is not meant to be a theory of hidden variables. This fact can be reflected from the fact that incompatible families of histories cannot be both interpreted as true. This lead to a variety of interpretations of families of histories. A critical consideration that must be taken into account when considering interpretations is whether any quasiclassical domains can be identified and if such domains persist throughout the whole histories. These quasiclassical domains must also be consistent with the real world quasiclassical phenomenon, which set the criteria in choosing the initial and final state of the universe.

However, such descriptions need a clear definition of true events and a theory of IGUSes experience. Moreover, an agreement on the measure of classicality is required. There has been investigation on measures of classicality, or in reverse, measures of quantumness, for example see ref. [48].

The operator method of decoherent histories approach is most conveniently applied in situations that are similar to those in standard quantum mechanics. By construction, when decoherence is achieved, the probabilities of measurement results can be reproduced when appropriate histories are considered but the approach also allows consideration of closed systems. Furthermore, descriptions of correlated subsystems do not require any notion of action over distance, the dynamics of each subsystems are independent of each other. Nonetheless, the operator approach is limited by the neccessary existence of a pre-existing background time, rather than deriving time as an emergent phenomenon. Therefore, the operator method cannot provide descriptions of any systems in which time enters non-trivially.

The path integral formulation, on the other hand, can be generalised to situations in which a background spacetime geometry do not exist. The path integral formulation is most useful in tackling problems in which time do not enter trivially such as the arrival problems and quantum cosmology. It also sees a great potential in providing a framework within which theories of quantum gravity can be developed and investigating the emergence of time, for example see ref. [70].

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Appendices

A Boolean Algebra

A Boolean algebra is the generalisation of logical operations to generic sets. Logical operations consists of two elements, labelled 0 and 1, and three operations on the set $\{0, 1\}$: conjunction ($\wedge$), disjunction ($\vee$) and negation ($\neg$) defined by

$$1 \wedge 0 = 0 \wedge 1 = 0 \wedge 0 = 0,\ \ 1 \wedge 1 = 1 \tag{A.1a}$$

$$1 \vee 0 = 0 \vee 1 = 1 \vee 1 = 1,\ \ 0 \vee 0 = 0 \tag{A.1b}$$

$$\neg 1 = 0,\ \ \neg 0 = 1 \tag{A.1c}$$

A generic Boolean algebra is a set $\mathfrak{B}$ containing a zero element (in analogue to 0) and a unity element (in analogue to 1) and three operations, generalisations of conjunctions, disjunction and negation that satisfies Equations A.1, in addition to:

$$a \wedge b = b \wedge a\ ;\ a \vee b = b \vee a \tag{A.2a}$$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c\ ;\ a \vee (b \vee c) = (a \vee b) \vee c \tag{A.2b}$$

$$a \wedge 0 = 0 \wedge a = 0\ ;\ 1 \vee a = a \vee 1 = 1 \tag{A.2c}$$

$$a \wedge \neg a = 0\ ;\ a \vee \neg a = 1 \tag{A.2d}$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\ ;\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{A.2e}$$

$$a \vee (a \wedge b) = a\ ;\ a \wedge (a \vee b) = a \tag{A.2f}$$

Equations A.2a are called the *commutative laws*; A.2b are called the *associative laws*; A.2c are the *identity laws*; A.2d defines the *complementary* of any elements in $\mathfrak{B}$; A.2e are the *distributive laws*; A.2f are the *absorptive laws*.

B Derivation of Path Integral Method From Operator Method

Use the natural unit so that the Planck's constant $h = 1$. Given a general Hamiltonian $\hat{H}(\hat{p}, \hat{q})$ and a configuration space path $Q(t)$.
Partition the path by times

$$t_1 < t_2 < \cdots < t_n$$

Projections onto the configuration space coordinates $Q(t_i)$ are denoted by $\hat{P}_i \equiv |Q_i\rangle\langle Q_i|$. At any time t_i, an exhaustive and exclusive set of events can be given by $\{\hat{P}_i,\ \hat{\mathbb{I}} - \hat{P}_i\}$.

By definition in the operator method, the C-representation of the history $(\hat{P}_1, \hat{P}_2, \cdots, \hat{P}_n)$ is given by

$$\hat{C} = \hat{P}_n(t_n) \cdots \hat{P}_1(t_1) \tag{B.1}$$

This can be written in another way as

$$\hat{C} = e^{-i\hat{H}t_n}|Q_n\rangle \prod_{a=1}^{n-1} \langle Q_{a+1}|e^{-i\hat{H}(t_{a+1}-t_a)}|Q_a\rangle\langle Q_1|e^{i\hat{H}t_1} \tag{B.2}$$

where the reference time is set to be $t = 0$.

Therefore at the center of the computation of C-representation is the amplitude $\langle Q_{a+1}|e^{-i\hat{H}(t_{a+1}-t_a)}|Q_a\rangle$. This can be evaluated by inserting

$$\mathbb{I} = \int \frac{d^d p_a}{(2\pi)^d} |p_a\rangle\langle p_a| \tag{B.3}$$

where d is the dimension of the configuration space, so that

$$\langle Q_{a+1}|e^{-i\hat{H}(t_{a+1}-t_a)}|Q_a\rangle = \int \frac{d^d p_a}{(2\pi)^d} \langle Q_{a+1}|p_a\rangle\langle p_a|e^{-i\hat{H}(t_{a+1}-t_a)}|Q_a\rangle \tag{B.4}$$

Using the relation

$$\langle Q_{a+1}|p_a\rangle = e^{iQ_{a+1}p_a} \tag{B.5}$$

and $\langle p|\hat{H}(\hat{p},\hat{Q})|Q\rangle = H(p,Q)$, Equation (B.4) is equivalent to

$$\langle Q_{a+1}|e^{-i\hat{H}(t_{a+1}-t_a)}|Q_a\rangle = \int \frac{d^d p_a}{(2\pi)^d} e^{i(p_a \frac{Q_{a+1}-Q_a}{t_{a+1}-t_a} - H(p_a,Q_a))(t_{a+1}-t_a)} \tag{B.6}$$

Putting this back into Equation (B.2):

$$\hat{C} = e^{-i\hat{H}t_n}|Q_n\rangle \left(\prod_{a=1}^{n-1}\int \frac{d^d p_a}{(2\pi)^d}\right) e^{\sum\limits_{a=1}^{n-1} i(p_a \frac{Q_{a+1}-Q_a}{t_{a+1}-t_a} - H(p_a,Q_a))(t_{a+1}-t_a)} \langle Q_1|e^{i\hat{H}t_1} \tag{B.7}$$

Therefore, taking inner product with $e^{-i\hat{H}t_f}|Q_f\rangle$ on the left and $e^{-i\hat{H}t_i}|Q_i\rangle$ on the right ($t_i = 0$ in this case), where Q_i and Q_f are the initial and final configuration space coordinates of the system, one recover the matrix element of C-representation in the limit of $n \to \infty$ as a momentum path integral:

$$\langle Q_f(t_f)|\hat{C}|Q_i(t_i)\rangle = \int \mathcal{D}p \; e^{i\int dt\, p\dot{Q} - H(p,Q)} \tag{B.8}$$

If the Hamiltonian is quadratic in momentum, the integral can be computed as a Gaussian integral. In such case one recovers Equation (2.17)

$$\langle Q_f(t_f)|\hat{C}|Q_i(t_i)\rangle = e^{iS[Q]} \tag{B.9}$$

The decoherence functional for completely fine grained configuration space path then follows from Equation (2.12):

$$D(Q',Q) = e^{i(S[Q]-S[Q'])}\delta_F(Q_f - Q'_f)\rho(Q_i,Q'_i) \tag{B.10}$$

The δ-function here is the functional delta function and $\rho(Q_i,Q'_i)$ is the matrix element of the initial state $\langle Q_i|\hat{\rho}|Q'_i\rangle$.